

Gap generation for Dirac fermions on Lobachevsky plane in a magnetic field

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We study symmetry breaking and gap generation for fermions in the 2D space of constant negative curvature (the Lobachevsky plane) in an external covariantly constant magnetic field in a four-fermion model. It is shown that due to the magnetic and negative curvature catalysis phenomena the critical coupling constant is zero and there is a symmetry breaking condensate in the chiral limit even in free theory. We analyze solutions of the gap equation in the cases of zero, weak, and strong magnetic fields. As a byproduct we calculate the density of states and the Hall conductivity for noninteracting fermions that may be relevant for studies of graphene.

I. INTRODUCTION

Dynamical symmetry breaking (DSB) and a mass (gap) generation for fermions usually requires the presence of a strong attractive interaction [1] in order to break symmetry that makes the quantitative study of DSB a difficult problem. Therefore, it is very interesting to consider the cases where DSB takes place in the regime of weak coupling. Three such examples are known.

The first is symmetry breaking in the presence of the Fermi surface (i.e. chemical potential of the system is nonzero). According to the Bardeen–Cooper–Schrieffer theory of superconductivity [2] (or the QCD color superconductivity studies at finite baryon density [3]), the Fermi surface is crucial for the formation of a bound state and a symmetry breaking condensate in the case of arbitrary small attraction between fermions. Indeed, according to [4], the renormalization group scaling in this case is connected only with the direction perpendicular to the Fermi surface. Therefore, the effective dimension of spacetime is $1 + 1$ from the viewpoint of renormalization group scaling. Since a bound state forms for arbitrary small attraction in $1 + 1$ dimension, this implies that the critical coupling constant is zero in this case.

The second example of DSB in the regime of weak coupling is DSB in a constant magnetic field [5, 6] (for a short review see [7]), where symmetry is again dynamically broken for arbitrary weak interaction. The physical reason for this is the effective dimensional reduction of spacetime for fermions in the infrared region by 2 units in a constant magnetic field (DSB in a magnetic field in spacetimes of dimension higher than four was considered in [8]). The reduction occurs because electrons being charged particles cannot propagate in directions perpendicular to the magnetic field when their energy is much less than the Landau gap $\sqrt{|eB|}$.

Dynamics of fermions in hyperbolic spaces H^D gives the third known example of DSB with zero critical coupling constant [9] (for an excellent review of DSB in curved spacetime see [10]). Analyzing the heat kernel, it was shown in [11] that the zero value of the critical coupling constant for DSB in hyperbolic spaces is connected with an effective dimensional reduction in the infrared region for fermions. The combined effect of constant magnetic field and negative curvature of spacetime on the dynamics of symmetry breaking was studied in [12], where magnetic field was treated exactly but gravitational field was considered in the weak curvature approximation.

In this paper, we investigate DSB and gap generation for fermions in the $R \times H^2$ spacetime of constant negative curvature (the Lobachevsky plane) with an external covariantly constant magnetic field by treating gravitational and magnetic fields exactly. The study of effects of surface curvature may be important for some condensed matter systems [13, 14, 15]. In particular, the quantum Hall effect (QHE) for two-dimensional nonrelativistic electron gas on the surface of constant negative curvature was studied in Ref.[13]. The present work may be relevant for the QHE in graphene [14] (see also [16]) whose quasiparticle excitations possess the linear dispersion law and are described by the Dirac equation (with the light velocity replaced by the Fermi velocity). Although the Lobachevsky plane has not been experimentally realized yet, the interaction with impurities or defects in graphene may lead to the effective Dirac equation in a curved space. For example, substitution of some hexagons by pentagons (heptagons) in the hexagonal lattice leads to the warping of the graphene sheet and induces positive (negative) curvature in it [15]. The curvature of the graphene sheet influences the density of states and affects the transport properties (the QHE among them) that can be observed in experiment.

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In Sec.II we present a general expression for the effective potential in a four-fermion model of the Gross-Neveu type on the Lobachevsky plane in a constant magnetic field. We then analyze the dynamics of free fermions and calculate the density of states and the Hall conductivity. In particular, we show that the anomalous half-integer QHE takes place for the Lobachevsky plane and the effect of nonzero curvature is to shift the plateau transitions in the Hall conductivity to higher values of magnetic field and decrease the plateau widths. The gap equation for interacting fermions is derived in Sec.III and its analysis is given in subsections III A, III B, III C. We find solutions of the gap equation in the cases of zero, weak, and strong magnetic fields. For the case of zero external magnetic field, using solutions of the Dirac equation and studying classical motion in H^D , we clarify in Appendix physical reasons for the effective dimensional reduction $D + 1 \rightarrow 1 + 1$ in the infrared region for fermions in spacetimes $R \times H^D$.

II. FOUR-FERMION MODEL ON THE LOBACHEVSKY PLANE: GENERAL ANALYSIS

The Lobachevsky plane (or 2D hyperbolic space H^2) is the simplest example of a space with constant negative curvature. Using Poincare coordinates for the H^2 space, the interval on the static spacetime $R \times H^2$ is given by

$$ds^2 = dt^2 - \frac{a^2}{y^2}(dx^2 + dy^2), \quad (1)$$

where $y > 0$ and a is the curvature radius of the Lobachevsky space. The vector potential $\vec{A} = (\frac{Ba^2}{y}, 0)$ defines covariantly constant magnetic field on the Lobachevsky plane. Indeed, nonzero components of the strength tensor are

$$F_{12} = -F_{21} = \frac{Ba^2}{y^2},$$

and one can easily check that the corresponding $F_{\mu\nu}$ is covariantly constant, i.e., $\nabla^\mu F_{\mu\nu} = 0$.

We consider a four-fermion model of the Gross-Neveu type with N_f flavors in $2 + 1$ spacetime whose action reads

$$S = \int d^3x \sqrt{-g} \left[\sum_{k=1}^{N_f} \bar{\psi}_k i \gamma^\mu \nabla_\mu \psi_k + \frac{G}{2N_f} \left(\sum_{k=1}^{N_f} \bar{\psi}_k \psi_k \right)^2 \right], \quad (2)$$

where $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor, $\nabla_\mu = \partial_\mu + ieA_\mu + i\omega_\mu^{ab}\sigma_{ab}$ the covariant derivative with the spin connection ω_μ^{ab} , and γ^μ matrices in curved spacetime are related to the Dirac γ^a matrices in flat spacetime through dreibeines $\gamma^\mu = e_a^\mu \gamma^a$, and we use a reducible, four-dimensional representation of the Dirac algebra. Model (2) (with $N_f = 2$) in flat space was recently proposed as the low-energy theory of interacting electrons on graphene's two-dimensional honeycomb lattice [17] where a four-fermion term arises from the microscopic lattice interactions.

The action (2) is invariant with respect to $U(1) \times U(1)$ continuous transformations $\psi \rightarrow e^{i\alpha}\psi$, $\psi \rightarrow e^{i\theta\gamma_3\gamma_5}\psi$, and the discrete chiral transformations

$$\psi \rightarrow -i\gamma_3\psi, \quad \psi \rightarrow \gamma_5\psi. \quad (3)$$

The mass term $m\bar{\psi}\psi$ would break these discrete symmetries while keeping intact the continuous symmetries.

It is convenient to use the auxiliary field method and represent the action (2) in the equivalent form

$$S = \int d^3x \sqrt{-g} \left[\sum_{k=1}^{N_f} (i\bar{\psi}_k \gamma^\mu \nabla_\mu \psi_k - \sigma \bar{\psi}_k \psi_k) - \frac{N_f}{2G} \sigma^2 \right], \quad (4)$$

where σ is an auxiliary field. If the field $\sigma(x)$ acquires a nonzero vacuum expectation value, then fermions obtain nonzero mass and the flavor symmetries connected with the discrete $-i\gamma_3$ and γ_5 transformations are spontaneously broken. To find the effective action for the field $\sigma(x)$, we integrate over the fermion fields in the functional integral. We obtain

$$\Gamma(\sigma) = - \int d^3x \sqrt{-g} \frac{N_f \sigma^2}{2G} - i \text{Ln Det}(i\gamma^\mu \nabla_\mu - \sigma(x)). \quad (5)$$

The effective potential $V(\sigma)$ is evaluated for constant field configurations ($\sigma(x) = \text{const}$) and is given by the expression

$$V(\sigma) = - \frac{\Gamma(\sigma)}{\int d^3x \sqrt{-g}}.$$

Since

$$\text{Det}(i\gamma^\mu \nabla_\mu - \sigma) = \text{Det}[\gamma_5(i\gamma^\mu \nabla_\mu - \sigma)\gamma_5] = \text{Det}(-i\gamma^\mu \nabla_\mu - \sigma), \quad (6)$$

we find

$$\text{Ln Det}(i\gamma^\mu \nabla_\mu - \sigma) = \frac{1}{2} \text{Tr} \text{Ln}(\partial_0^2 + D^2 + \sigma^2) = -\frac{1}{2} \int_0^\infty \frac{dt}{t} \text{Tr} \exp[-i(\partial_0^2 + D^2 + \sigma^2)t],$$

where $D = \vec{\gamma} \vec{\nabla}$, the trace Tr is taken in the functional sense and we used the formal identity $\ln(H - i\epsilon) = -\int_0^\infty \exp[-it(H - i\epsilon)]dt/t$ for the logarithm of an operator H . Further we calculate

$$\text{Ln Det}(i\gamma^\mu \nabla_\mu - \sigma) = -\frac{1}{2} \int_{-\infty}^\infty \frac{dk_0}{2\pi} \int_0^\infty \frac{dt}{t} e^{ik_0^2 t} \text{Tr} e^{-i(D^2 + \sigma^2)t} = \frac{-iN_f}{2(4\pi)^{1/2}} \int d^3x \sqrt{-g} \int_0^\infty \frac{ds}{s^{3/2}} \text{tr} \langle \mathbf{x} | e^{-s(D^2 + \sigma^2)} | \mathbf{x} \rangle, \quad (7)$$

where the trace tr goes over the Dirac indices and in the second equality we deformed the integration contour $t \rightarrow -is$.

The gap equation $dV/d\sigma = 0$ determines a dynamically generated mass of fermions but before analyzing it we consider the dynamics of free fermions on the Lobachevsky plane in the presence of an external magnetic field.

A. Free fermions on the Lobachevsky plane

The energy spectrum and eigenfunctions of the Dirac operator on the Lobachevsky plane in an external covariantly constant magnetic field were found by Comtet and Houston [18]. Comparing with the Landau problem in flat space, the nonzero curvature of the Lobachevsky plane qualitatively changes the energy spectrum which consists of the discrete part (we assume $eB > 0$)

$$E_n = \pm \sqrt{\sigma^2 + \frac{b^2 - (n-b)^2}{a^2}}, \quad (8)$$

where $n = 0, 1, \dots$, $0 \leq n < b$, $b = eBa^2$, and the continuum part

$$E_\nu = \pm \sqrt{\sigma^2 + \frac{b^2 + \nu^2}{a^2}}, \quad (9)$$

where $0 \leq \nu < \infty$. In the limit $a \rightarrow \infty$, the continuum part of the spectrum disappears (goes to infinity) and $E_n \rightarrow \pm \sqrt{\sigma^2 + 2neB}$, i.e. the spectrum coincides with the spectrum of the Landau problem in flat space.

The eigenfunctions of the discrete spectrum are

$$\begin{aligned} \psi_{0,k}^{(d)}(x, y) &= \frac{e_-}{a} f_0(k, b + \frac{1}{2}; x, y), \quad k < 0, \\ \psi_{\alpha,n,k}^{(d)}(x, y) &= \frac{e_\alpha}{a} f_{n+\frac{\alpha-1}{2}}(k, b - \frac{\alpha}{2}; x, y), \quad k < 0, \quad n = 1, 2, \dots, \quad 1 \leq n < b. \end{aligned} \quad (10)$$

Here α takes values $+1$ and -1 and $\{e_+, e_-\}$ consists of orthonormal constant eigenvectors of $\gamma^1 \gamma^2$: $\gamma^1 \gamma^2 e_\pm = \pm i e_\pm$, and the functions f_n are

$$f_n(k, \beta; x, y) = \sqrt{\frac{n!(2\beta - 2n - 1)}{4\pi|k|\Gamma(2\beta - n)}} e^{-ikx} e^{-|k|y} (2|k|y)^{\beta-n} L_n^{(2\beta-2n-1)}(2|k|y), \quad (11)$$

where $L_n^{(\alpha)}(z)$ denote associated Laguerre polynomials [19]. Note that for fixed k , the lowest level of the discrete spectrum is not degenerate unlike the higher levels with $n \geq 1$ which are twice degenerate, similar to the case of spectrum of the Landau problem in flat space.

The eigenfunctions of the continuum spectrum are

$$\psi_{\alpha,\nu,k}^{(c)}(x, y) = \frac{e_\alpha}{a} f(\nu, k, |b - \frac{\alpha}{2}|; x, y), \quad k(b - \frac{\alpha}{2}) > 0, \quad (12)$$

where

$$f(\nu, k, \beta; x, y) = \sqrt{\frac{\nu \sinh(2\pi\nu)}{4\pi^3|k|}} |\Gamma(i\nu - \beta + \frac{1}{2})| e^{-ikx} z^{1/2} e^{-z/2} \left[\frac{\Gamma(i\nu)}{\Gamma(\frac{1}{2} + i\nu - \beta)} z^{-i\nu} M(\frac{1}{2} - i\nu - \beta, 1 - 2i\nu; z) \right. \\ \left. + \frac{\Gamma(-i\nu)}{\Gamma(\frac{1}{2} - i\nu - \beta)} z^{i\nu} M(\frac{1}{2} + i\nu - \beta, 1 + 2i\nu; z) \right], \quad z = 2|k|y, \quad (13)$$

and $M(a, b; z)$ is the confluent hypergeometric function. Using the above eigenfunctions the heat kernel of the operator $D^2 + \sigma^2$ in (7) is calculated to be (see Ref.[18]),

$$\text{tr} \langle \mathbf{x} | e^{-s(D^2 + \sigma^2)} | \mathbf{x} \rangle = \frac{1}{\pi a^2} \left\{ b e^{-s\sigma^2} + 2 \sum_{n=1}^{[b]} (b-n) e^{-s(\sigma^2 + \frac{2nb-n^2}{a^2})} \right. \\ \left. + \frac{1}{\pi} \int_0^\infty d\nu \nu e^{-s(\sigma^2 + \frac{b^2 + \nu^2}{a^2})} \text{Im} [\psi(i\nu - b) + \psi(i\nu - b + 1) + \psi(i\nu + b + 1) + \psi(i\nu + b)] \right\}, \quad (14)$$

where $[b]$ is the largest integer satisfying $[b] \leq b$ (note that our expression differs from that in [18] by factor 2 because we consider a reducible representation of the Dirac algebra). The series representation for the imaginary part of ψ -function [19] yields

$$\text{Im} [\psi(i\nu - b) + \psi(i\nu - b + 1) + \psi(i\nu + b + 1) + \psi(i\nu + b)] = \sum_{k=-\infty}^{+\infty} \frac{2\nu}{\nu^2 + (k-b)^2}. \quad (15)$$

and the sum in Eq.(15) is evaluated to be

$$\sum_{k=-\infty}^{+\infty} \frac{2\nu}{\nu^2 + (k-b)^2} = \frac{2\pi \sinh(2\pi\nu)}{\cosh(2\pi\nu) - \cos(2\pi b)}. \quad (16)$$

Finally, we obtain

$$\text{tr} \langle \mathbf{x} | e^{-s(D^2 + \sigma^2)} | \mathbf{x} \rangle = \frac{1}{\pi a^2} \left[b e^{-s\sigma^2} + 2 \sum_{n=1}^{[b]} (b-n) e^{-s(\sigma^2 + \frac{2nb-n^2}{a^2})} + 2 \int_0^\infty d\nu \nu e^{-s(\sigma^2 + \frac{b^2 + \nu^2}{a^2})} \frac{\sinh(2\pi\nu)}{\cosh(2\pi\nu) - \cos(2\pi b)} \right]. \quad (17)$$

B. Density of states and Hall conductivity

Using the fermion Green's function

$$G(\mathbf{x}, \mathbf{x}'; E + i\epsilon) = \langle \mathbf{x} | \frac{1}{\gamma^0(E + i\epsilon) - i\vec{\gamma}\vec{D} - \sigma} | \mathbf{x}' \rangle,$$

we calculate the density of states (DOS) for noninteracting theory which is defined as

$$\rho(E) = -\frac{N_f}{\pi V} \int d^2x \text{tr} [\gamma^0 \text{Im} G(\mathbf{x}, \mathbf{x}; E + i\epsilon)] = \frac{N_f}{\pi} \text{Im} \text{tr} \langle \mathbf{x} | \frac{E}{D^2 + \sigma^2 - (E + i\epsilon)^2} | \mathbf{x} \rangle, \quad (18)$$

where V is space volume which is canceled because diagonal matrix elements of the operator D^2 do not depend on \mathbf{x} (see Eq.(17)) in view of homogeneity of the Lobachevsky space. In order to calculate the DOS we integrate Eq.(17) over s from 0 to ∞ , replace σ^2 by $\sigma^2 - (E + i\epsilon)^2$, and take the imaginary part. We get

$$\rho(E) = \frac{N_f e B}{2\pi} [\delta(E - \sigma) + \delta(E + \sigma)] + \frac{N_f}{\pi a^2} \sum_{n=1}^{[b]} (b-n) [\delta(E - E_n) + \delta(E + E_n)] \\ + \frac{N_f |E|}{\pi} \theta \left(|E| - \sqrt{\sigma^2 + \frac{b^2}{a^2}} \right) \frac{\sinh(2\pi\nu(E))}{\cosh(2\pi\nu(E)) - \cos(2\pi b)}, \quad (19)$$

where

$$E_n = \sqrt{\sigma^2 + \frac{2nb - n^2}{a^2}}, \quad \nu(E) = \sqrt{a^2(E^2 - \sigma^2) - b^2}. \quad (20)$$

The first two terms in Eq.(19) correspond to the point spectrum and the last term corresponds to the continuous one.

One can easily check that in flat space ($a = \infty$) the DOS given by Eq.(19) reduces to the corresponding expression (4.2) in Ref.[20]. On the other hand, in the absence of a magnetic field ($B = 0$) we have

$$\rho(E) = \frac{N_f |E|}{\pi} \theta(|E| - \sigma) \coth(\pi a \sqrt{E^2 - \sigma^2}). \quad (21)$$

For massless fermions ($\sigma \rightarrow 0$) we find that the DOS (21) remains finite at $E = 0$,

$$\rho(0) = \frac{N_f}{\pi^2 a}, \quad (22)$$

in contrast to the flat space where it vanishes like $\rho(E) \sim |E|$ when $E \rightarrow 0$. It also differs from nonrelativistic fermions on the Lobachevsky plane where the DOS behaves as $\rho(E) \sim \sqrt{|E|}$ when $E \rightarrow 0$ [13].

In the case of high magnetic fields and large radius of curvature ($e^2 B^2 a^2 > \mu^2 - \sigma^2$) the energy spectrum below the Fermi level μ is only the discrete one, therefore the last term in Eq.(19) related to the continuum spectrum does not contribute. In this case the number of states below the Fermi level $N(\mu)$ can be easily calculated using Eq.(5.23) at zero temperature in Ref.[20],

$$N(\mu) = V \text{sign}(\mu) \int_0^{|\mu|} dE \rho(E) = \frac{N_f V}{2\pi a^2} \text{sign}(\mu) \left(b + 2(b - \frac{n_{max} + 1}{2}) n_{max} \right), \quad (23)$$

where n_{max} equals the maximal filled state number and is given by the integer part of the following expression:

$$n_{max} = [b - \sqrt{b^2 - a^2(\mu^2 - \sigma^2)}].$$

Using $N(\mu)$, we can calculate the Hall conductivity through the Streda formula [21],

$$\sigma_{xy}(\mu, B) = -\frac{e}{V} \frac{\partial N(\mu)}{\partial B}, \quad e > 0, \quad (24)$$

which is valid when the Fermi level lies within an energy gap. In the energy gap the integer part of $b - \sqrt{b^2 - a^2(\mu^2 - \sigma^2)}$ is constant and we obtain

$$\sigma_{xy}(\mu, B) = -\text{sign}(\mu) \frac{N_f e^2}{\pi} \left(\frac{1}{2} + [b - \sqrt{b^2 - a^2(\mu^2 - \sigma^2)}] \right). \quad (25)$$

This expression describes the Hall conductivity of relativistic fermions on the Lobachevsky plane. As is seen, the field dependence of the Hall conductivity has a familiar step-like behavior. In the limit of zero curvature ($a \rightarrow \infty$) we get

$$\sigma_{xy}(\mu, B) = -\text{sign}(\mu) \frac{N_f e^2}{\pi} \left(\frac{1}{2} + [\frac{\mu^2 - \sigma^2}{2eB}] \right). \quad (26)$$

For massless fermions, $\sigma = 0$, this expression coincides (for $N_f = 2$ and restoring the constants \hbar, c and the Fermi velocity v_F) with Eq.(7) in Ref.[14] which describes the unconventional quantum Hall effect in graphene. The anomalous QHE with half-integer quantization of the Hall conductivity for Dirac fermions is due to the lowest Landau level whose degeneracy is two times less than degeneracy of any other level. Notice that this anomalous QHE is the most direct evidence for the existence of Dirac fermions in graphene [22]. The half-integer quantization of the Hall conductivity remains valid for the Lobachevsky plane (for the QHE exhibited by relativistic particles on a two-sphere see recent paper [23]). The effect of nonzero negative curvature is to shift the plateau transitions (which arise from the crossings of the Fermi level with the Landau levels) in the Hall conductivity to higher magnetic fields,

$$B_n = \frac{\mu^2 - \sigma^2}{2en} + \frac{n}{2ea^2}, \quad (27)$$

and decrease the plateau widths,

$$\Delta B = \frac{\mu^2 - \sigma^2}{2en(n+1)} - \frac{1}{2ea^2}, \quad (28)$$

similar to the case of nonrelativistic electrons [13].

III. GAP EQUATION

We now turn to the analysis of the gap equation $dV/d\sigma = 0$ which takes the following form:

$$\sigma = \frac{G}{\pi a^2} \left[\frac{b}{2} + \sum_{n=1}^{[b]} \frac{b-n}{\sqrt{1 + \frac{2nb-n^2}{\sigma^2 a^2}}} + \int_0^{\Lambda a} \frac{d\nu \nu}{\sqrt{1 + \frac{b^2 + \nu^2}{\sigma^2 a^2}}} \frac{\sinh(2\pi\nu)}{\cosh(2\pi\nu) - \cos(2\pi b)} \right], \quad (29)$$

where Λ is the ultraviolet (UV) cut-off. In what follows we find approximate analytical solutions of the gap equation (29) in the cases $b = 0$, $b \ll 1$, and $b \gg 1$.

It is instructive before solving the gap equation to calculate the symmetry breaking condensate $\langle 0|\bar{\psi}\psi|0 \rangle$ in the chiral limit $\sigma \rightarrow 0$. Naively one would expect that it is zero in the chiral limit. However, like in the case of free fermions in flat $(2+1)$ -dimensional spacetime with $B \neq 0$ [5] the condensate is nonzero in the chiral limit in the case under consideration. The reason is that, although the Landau spectrum is modified for fermions in a covariantly constant magnetic field in H^2 (see [18]), the lowest zero level still survives and this leads to the effective reduction of spacetime dimension by two units and, as result, to a nonzero condensate. The condensate in our model equals

$$\langle 0|\bar{\psi}\psi|0 \rangle = - \lim_{x \rightarrow x'} \text{tr} G(x, x') = - \frac{N_f \sigma}{2\sqrt{\pi}} \int_0^\infty \frac{ds}{\sqrt{s}} \text{tr} \langle \vec{x} | e^{-s(D^2 + \sigma^2)} | \vec{x} \rangle, \quad (30)$$

where $G(x, x')$ is the fermion Green's function. Using the heat kernel Eq.(14), we find that only the lowest Landau level contributes to the condensate in the chiral limit $\sigma \rightarrow 0$:

$$\langle 0|\bar{\psi}\psi|0 \rangle = -N_f \sigma \int_0^\infty \frac{ds}{\sqrt{4\pi s}} \frac{b e^{-s\sigma^2}}{\pi a^2} = -\frac{N_f e B}{2\pi}. \quad (31)$$

Note that condensate (31) does not depend on curvature of spacetime and exactly coincides with the corresponding flat spacetime result [5].

A. Zero Magnetic Field

In this section we consider solutions of the gap equation for zero magnetic field making an accent on the physics underlying the phenomenon of spontaneous mass generation in hyperbolic spaces. Unlike the dimensional regularization usually considered in the literature, we use the regularization with an explicit UV cut-off.

For $B = 0$, the gap equation (29) significantly simplifies and we get

$$\sigma = \frac{G\sigma}{\pi a} \int_0^{\Lambda a} \frac{d\nu \nu}{\sqrt{\sigma^2 a^2 + \nu^2}} \coth(\pi\nu), \quad (32)$$

which up to the terms of order $1/\Lambda$ can be rewritten in the form

$$\sigma = \frac{G\sigma}{\pi} (\Lambda - \sigma) + \frac{G\sigma}{\pi^2 a} \int_0^\infty \frac{d\nu \nu (\coth \nu - 1)}{\sqrt{\nu^2 + (\pi\sigma a)^2}}. \quad (33)$$

As is seen, there always exists the trivial solution $\bar{\sigma} = 0$. In the case of flat space ($a = \infty$) the integral on the right hand side of Eq.(33) vanishes and the equation admits a nontrivial solution only if the coupling constant G is supercritical, $G > G_c = \pi/\Lambda$. However, on the Lobachevsky plane with the finite curvature radius a the situation changes dramatically: a nontrivial solution exists for all $G > 0$. The reason for this is that in the Lobachevsky space the interaction becomes enhanced in the infrared region (small ν): the integral in Eq.(33) is proportional to $\ln(1/(\pi\sigma a))$ as $\sigma \rightarrow 0$. Analytical solution can be obtained for the coupling constant $G \ll a$. We find

$$m_{dyn} \equiv \bar{\sigma} = \frac{1}{\pi a} \exp\left(-\frac{\pi^2 a}{G}\right). \quad (34)$$

Note that this solution does not depend on the ultraviolet cutoff Λ . This confirms that the effect of mass generation in the Lobachevsky space in the weak coupling regime is of purely infrared origin.

To study the case of a strong coupling we introduce the dimensionless coupling $G = \pi g/\Lambda$ and the scale $m^* = \Lambda(1/g_c - 1/g)$ where we defined the critical coupling in the flat space $g_c = 1$. Then Eq.(33) is rewritten in the form

$$\pi m^* a = \pi \sigma a - \int_0^\infty \frac{d\nu \nu (\coth \nu - 1)}{\sqrt{\nu^2 + (\pi \sigma a)^2}}. \quad (35)$$

In the near critical region $g - g_c \ll 1/\Lambda a$ the scale $m^* \simeq 0$ and the dynamical mass is given by the root of the right hand side of Eq.(35),

$$m_{dyn} = \bar{\sigma} = \frac{0.8}{\pi a}. \quad (36)$$

Thus in the scaling region $g - g_c \ll 1/\Lambda a$, the cutoff disappears from the observable quantity m_{dyn} . The critical value $g_c = 1$ is in fact an UV stable fixed point at the leading order in $1/N_f$ expansion and the relation (36) can be considered as a scaling law in the scaling region.

In the supercritical region $g > g_c$, the analytical expression for m_{dyn} can be obtained at large curvature radius a , satisfying the condition $m^* a \gg 1$ (note that m^* is the solution of the gap equation (35) for $a \rightarrow \infty$). Expanding the integral on the right hand side of Eq.(35) in $1/\pi \sigma a$ we find

$$m_{dyn} = \bar{\sigma} = m^* \left(1 + \frac{1}{12(m^* a)^2} \right), \quad (37)$$

i.e., m_{dyn} increases with the decrease of the curvature radius a . In fact a numerical study of Eq.(35) shows that the dynamical mass m_{dyn} increases with the decrease of a for all values of g and a .

It is instructive to compare solution (34) with the relation for the dynamical mass in the $(1+1)$ -dimensional Gross-Neveu (GN) model [24] and with the quasiparticle gap in the BCS theory of superconductivity [2]. The relation for the dynamical mass in the Gross-Neveu model is

$$m_{dyn} = \Lambda \exp \left(-\frac{\pi}{N_f G^{(0)}} \right), \quad (38)$$

where $G^{(0)}$ is the bare coupling, which is dimensionless for $D = 1 + 1$. The similarity between Eqs.(34) and (38) is evident: $1/\pi a$ and $G/\pi a$ in Eq.(34) play the role of an ultraviolet cutoff and the dimensionless coupling constant in Eq.(38), respectively. This reflects the point that the dynamics of fermion pairing in the Lobachevsky space is essentially $(1+1)$ -dimensional.

We recall that in the theory of superconductivity due to the presence of the Fermi surface the dynamics of electrons is also effectively $(1+1)$ -dimensional. The analogy with the superconductivity theory is even deeper than with the GN model. Indeed, the energy gap in the BCS theory has the form $\Delta \sim \omega_D \exp(-const/\nu_S G_S)$, where ω_D is the Debye frequency, G_S is a coupling constant and ν_S is the density of states on the Fermi surface. In the present model $\rho(E=0) = N_f/\pi^2 a = \nu_0$, where ν_0 is the density of states on the energy surface $E=0$ (see Eq.(22)). Thus the energy surface $E=0$ plays here the role of the Fermi surface. Hence Eq.(34) can be rewritten in the form $m_{dyn} = (1/\pi a) \exp(-N_f/\nu_0 G)$ exhibiting a complete analogy with the energy gap in the BCS theory. Moreover, it can be shown that the effective dimensional reduction $D+1 \rightarrow 1+1$ for fermions in the infrared region takes place for $R \times H^D$ spacetimes of any dimension $D \geq 2$ and the same reduction $D+1 \rightarrow 1+1$ is valid for the BCS theory in $D+1$ dimension. In Appendix we present physical reasons for the reduction $D+1 \rightarrow 1+1$ in hyperbolic spacetimes $R \times H^D$ studying the classical motion of particles and solving also the Dirac equation in these spacetimes.

B. Weak Magnetic Field

To study the dynamical mass generation in the Lobachevsky space in the presence of an external magnetic field we first rewrite Eq.(29) in more convenient form,

$$\sigma = \frac{G\sigma}{\pi}(\Lambda - \sigma) + \frac{G\sigma}{\pi a} \left[\frac{b}{2\sigma a} + \sum_{n=1}^{[b]} \frac{b-n}{\sqrt{(\sigma a)^2 + 2bn - n^2}} + \int_0^\infty \frac{d\nu \nu}{\sqrt{\nu^2 + b^2 + (\sigma a)^2}} \left(\frac{\sinh(2\pi\nu)}{\cosh(2\pi\nu) - \cos(2\pi b)} - 1 \right) \right], \quad (39)$$

up to terms of order $1/\Lambda$. For a weak external magnetic field when $b \ll \min(1, \sigma a)$ or equivalently the magnetic length $l = 1/\sqrt{eB}$ satisfying $l \gg \max(a, \sqrt{\frac{a}{\sigma}})$, we neglect b^2 and higher order terms in Eq.(39) and obtain the

following gap equation:

$$\pi\sigma a = \pi m^* a + \frac{\pi b}{2\sigma a} + \int_0^\infty \frac{d\nu \nu (\coth \nu - 1)}{\sqrt{\nu^2 + (\pi\sigma a)^2}}, \quad (40)$$

where the second term on the right hand side represents, obviously, a first order correction to the gap equation (35). Seeking the solution in the form $\sigma = m_{dyn}^{(0)} + Cb$ where $m_{dyn}^{(0)}$ is the solution of the zero field gap equation (35), we find

$$m_{dyn} \equiv \bar{\sigma} = m_{dyn}^{(0)} \left(1 + \frac{eB}{2(m_{dyn}^{(0)})^2} \right), \quad (41)$$

i.e., the dynamical mass always increases with B . A striking fact is that, unlike the gap equation (33) with $B = 0$, the gap equation (39) with $B \neq 0$ does not have the trivial solution $\bar{\sigma} = 0$. Thus, despite the spontaneous character of breaking of the discrete symmetries (3), there is no trivial solution in the magnetic field for all values of the coupling constant G , the fact already known in the case of a flat space [6].

C. Strong Magnetic Field

For strong magnetic field $b \gg 1$, we can determine the leading asymptotics of the sum in (39) as $b \rightarrow \infty$. We have

$$\sum_{n=1}^b \frac{b-n}{\sqrt{\sigma^2 a^2 + 2nb - n^2}} = b \sum_{n=1}^b \frac{1}{b} \frac{1 - \frac{n}{b}}{\sqrt{\frac{\sigma^2 a^2}{b^2} + \frac{2n}{b} - \frac{n^2}{b^2}}}. \quad (42)$$

For $b \rightarrow \infty$, we can make the change $\frac{1}{b} \rightarrow dx$ and replace the sum over n by integral over x . Then sum (42) is approximated by the integral

$$I = b \int_{\frac{1}{b}}^1 dx \frac{1-x}{\sqrt{\frac{\sigma^2 a^2}{b^2} + 2x - x^2}} = \sqrt{b^2 + \sigma^2 a^2} - \sqrt{2b - 1 + \sigma^2 a^2}.$$

Consequently, we obtain the following gap equation:

$$\sigma = \frac{G}{\pi a^2} \left[\frac{b}{2} + \sigma a I + \Lambda \sigma a^2 - \sigma a \sqrt{b^2 + (\sigma a)^2} + \sigma a \int_0^\infty \frac{d\nu \nu}{\sqrt{\nu^2 + b^2 + (\sigma a)^2}} \left(\frac{\sinh(2\pi\nu)}{\cosh(2\pi\nu) - \cos(2\pi b)} - 1 \right) \right]. \quad (43)$$

Let us now consider solutions of this gap equation for $\sigma a \ll b$ and $\sigma a \gg b$. For $\sigma a \ll b$, $I \simeq b - \sqrt{2b}$ and we can neglect the integral in Eq.(43) which is of order $1/b$. Thus, we find the solution

$$m_{dyn} = \bar{\sigma} = \frac{Gb/(2\pi a^2)}{1 - \frac{G\Lambda}{\pi} + \frac{G\sqrt{2eB}}{\pi}} \approx \frac{Gb/(2\pi a^2)}{1 - \frac{G\Lambda}{\pi}}. \quad (44)$$

This solution is obviously valid for $G < \pi/\Lambda$, i.e., this solution corresponds to the weak coupling regime. One can check that the condition $\sigma a \ll b$ is also satisfied because

$$m_{dyn} = \bar{\sigma} \approx \frac{Gb}{2\pi a^2} = \frac{GeB}{2\pi} \quad (45)$$

in the weak coupling regime. Solution (45) is exactly the flat spacetime solution in $2+1$ dimension in a constant external magnetic field [5]. Of course, this is a natural result because for strong magnetic field $eB \gg 1/a^2$, we can neglect small corrections due to the curvature of space. Further, one can check that solution (45) satisfies the condition $\bar{\sigma} a \ll b$ because $G \ll 2\pi a$.

For the other case $\sigma a \gg b$, $I \simeq b^2/(2\sigma a)$ and we find

$$\sigma \approx \frac{G}{\pi a^2} \left(\frac{b^2}{2} + \sigma \Lambda a^2 - \sigma^2 a^2 \right), \quad (46)$$

that gives

$$m_{dyn} = \bar{\sigma} = \frac{m^* + \sqrt{m^{*2} + \frac{2b^2}{a^2}}}{2}. \quad (47)$$

The condition $\sigma a \gg b$ implies that $m^{*2} = (\Lambda - \frac{\pi}{G})^2$ should be much larger than $2b^2/a^2$. Therefore, this solution exists for $G > \pi/\Lambda$, i.e. it is a strong coupling solution. Using the strong coupling solution $m^* = \Lambda - \frac{\pi}{G}$ in flat space ($a = \infty$), we can represent Eq.(47) as follows:

$$m_{dyn} \equiv \bar{\sigma} \approx m^* + \frac{b^2}{2m^{*2}a^2} = m^* + \frac{e^2 B^2 a^2}{2m^*}. \quad (48)$$

Thus, we conclude that for strong external magnetic field $b \gg 1$ the dynamical fermion mass coincides with the $2 + 1$ -dimensional flat spacetime solution (45) in the weak coupling regime and, according to Eq.(48), the correction to the strong coupling solution m^* due to external magnetic field is quadratic in B .

IV. CONCLUSION

In the present paper we studied a four-fermion Gross-Neveu type model on the Lobachevsky plane in an external covariantly constant magnetic field. For noninteracting fermions we calculated the density of states and the Hall conductivity which may be relevant for experimental investigations of graphene. In particular, we showed that the density of states for free massless fermions is finite at zero energy in contrast to the case of flat space where it vanishes. The anomalous half-integer quantization of the Hall conductivity remains valid for the Lobachevsky plane and the effect of nonzero negative curvature is to shift the plateau transitions in the Hall conductivity to higher magnetic fields and decrease the plateau widths.

Studying the dynamical symmetry breaking we showed that discrete symmetries of this model are always spontaneously broken, i.e. the critical coupling constant is zero. Moreover, we found that there is a symmetry breaking condensate even in noninteracting free theory in the chiral limit. These facts are consequences of the effective dimensional reduction for fermions in the infrared region to a $(0 + 1)$ -dimensional theory. It is interesting that the condensate does not depend on the value of curvature of the Lobachevsky plane and exactly coincides with its value in flat $(2 + 1)$ -dimensional space in an external constant magnetic field [5]. We analyzed the gap equation and found its solutions for the cases of zero, weak, and strong magnetic fields.

For zero magnetic field, we showed that the dynamical mass essentially depends on the radius of curvature of H^2 in the weak coupling regime ($G \ll \pi/\Lambda$). In the strong coupling regime ($G > \pi/\Lambda$), the dynamical mass is virtually independent of the curvature of space and practically coincides with the flat space solution up to corrections of order $1/(m^*a)^2$ (Eq.(37)). For weak magnetic field ($eB \ll 1/a^2$), the correction to the dynamical mass due to external magnetic field is linear in B and the dynamical mass grows with magnetic field more quickly in the weak coupling regime than in the strong coupling regime. For strong magnetic field ($eB \gg 1/a^2$), we found that up to negligible corrections, the dynamical mass coincides with the dynamical mass in flat spacetime in an external constant magnetic field in the weak coupling regime. In the strong coupling regime, the correction to the flat spacetime solution due to external magnetic field is quadratic in B .

It was shown in [11] that the zero value of the critical coupling constant for DSB in $R \times H^D$ spacetimes is due to the effective dimensional reduction $D + 1 \rightarrow 1 + 1$ for fermions in the infrared region which takes place for any $D \geq 2$. In order to clarify physical reasons for this reduction, we considered in Appendix solutions of the Dirac equation on $R \times H^D$, where we showed that due to the spherical and scale symmetries the initial Dirac problem is reduced to an effective $(1 + 1)$ -dimensional problem. Further, according to [25], if dimensional reduction in the infrared region is observed in a quantum problem, then classical motion should have a bounded character with respect to the coordinates over which the reduction takes place, i.e. the physical system should effectively be of a finite size with respect to these coordinates. Studying the classical motion on H^D , we showed in Appendix that this is indeed the case.

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APPENDIX A

The metric of static $R \times H^D$ spacetime is given by

$$ds^2 = dt^2 - \frac{a^2}{x_1^2}(dx_1^2 + dx_2^2 + \dots + dx_D^2), \quad x_1 > 0. \quad (\text{A1})$$

The Dirac equation in this spacetime when magnetic field is absent has the form

$$\left(i\gamma^0 \partial_0 + \frac{ix_1}{a} \gamma^1 \partial_1 + \dots + \frac{ix_1}{a} \gamma^D \partial_D - \frac{i(D-1)}{2a} \gamma^1 - m \right) \psi = 0. \quad (\text{A2})$$

As usual, it is more convenient to work with the second order differential equation, which is obtained multiplying (A2) by $i\hat{D} + m$

$$(-\partial_0^2 + \frac{(D-1)^2}{4a^2} - \frac{D-2}{a^2} x_1 \partial_1 + \frac{x_1^2}{a^2} (\partial_1^2 + \dots + \partial_D^2) - \frac{x_1}{a^2} \gamma^1 \gamma^2 \partial_2 - \frac{x_1}{a^2} \gamma^1 \gamma^3 \partial_3 - \dots - \frac{x_1}{a^2} \gamma^1 \gamma^D \partial_D - m^2) \psi = 0. \quad (\text{A3})$$

Obviously, we can seek solution in the form $\psi = \exp[-i\omega x_0 + ip_2 x_2 + \dots + ip_D x_D] f(x_1)$

$$\left[\omega^2 + \frac{(D-1)^2}{4a^2} - \frac{D-2}{a^2} x_1 \partial_1 + \frac{x_1^2 \partial_1^2}{a^2} - \frac{x_1^2}{a^2} (p_2^2 + \dots + p_D^2) - i \frac{x_1}{a^2} \gamma^1 \gamma^2 p_2 - i \frac{x_1}{a^2} \gamma^1 \gamma^3 p_3 - \dots - i \frac{x_1}{a^2} \gamma^1 \gamma^D p_D - m^2 \right] f(x_1) = 0. \quad (\text{A4})$$

Further, γ matrices are present only in the operator

$$-i \frac{x_1}{a^2} \gamma^1 \gamma^2 p_2 - i \frac{x_1}{a^2} \gamma^1 \gamma^3 p_3 - \dots - i \frac{x_1}{a^2} \gamma^1 \gamma^D p_D.$$

Since this is an Hermitian operator and its square $x_1^2(p_2^2 + \dots + p_D^2)/a^4$ is a unit matrix, it can be diagonalized and its eigenvalues are equal to $\sigma x_1 \sqrt{p_2^2 + \dots + p_D^2}/a^2$, where $\sigma = \pm$. Making the change of variable $z = \sqrt{p_2^2 + \dots + p_D^2} x_1$, we obtain

$$\left(\omega^2 + \frac{z^2}{a^2} (-1 + \partial_z^2) + \frac{(D-1)^2}{4a^2} - \frac{(D-2)z}{a^2} \partial_z - \frac{\sigma z}{a^2} - m^2 \right) f(z) = 0. \quad (\text{A5})$$

The absence of any dependence on p_2, \dots, p_D in this equation is remarkable because it means that energy does not depend on them, i.e., it is the same for any p_2, \dots, p_D . Eq.(A5) has the form of equation of a $(1+1)$ -dimensional problem and it is easy to find its spectrum $\omega = \pm \sqrt{\nu^2/a^2 + m^2}$, where ν takes values in $(0, +\infty)$. We would like to note that the effective dimensional reduction $D+1 \rightarrow 1+1$ for fermions on hyperbolic spaces H^D was observed in [11] by analyzing the heat kernel of Dirac operator on these spaces, however, physical reasons for this reduction remained unclear. Here, we see that this reduction is connected with the effective $(1+1)$ -dimensional form (A5) of the Dirac equation. From the mathematical viewpoint, the reduction $D+1 \rightarrow 1+1$ is related to the spherical and scale symmetries of the H^D metric written in the Poincare patch. Indeed, the spherical symmetry of the x_2, \dots, x_D part of metric (A1) reduces the dependence of energy on p_2, \dots, p_D to the dependence on the only invariant $p^2 = p_2^2 + \dots + p_D^2$ and then the symmetry of metric (A1) with respect to scale transformations $x_i \rightarrow \lambda x_i$ ($i = \overline{1, D}$) eliminates any dependence on p_2, \dots, p_D in Eq.(A5) for eigenfunctions.

Actually there exists a physically even more transparent way to show the occurrence of the effective dimensional reduction on H^D . According to [25], the effective dimensional reduction in the infrared region takes place in a quantum problem only when the corresponding classical motion has a bounded character with respect to the coordinates over which the dimensional reduction occurs. For example, let us consider the dimensional reduction in constant magnetic field. In this case, classically a charged particle moves on circular orbits in the plane perpendicular to the constant magnetic field. Since the radius of its orbit is proportional to energy, a charged particle can go to infinity only if it has infinite energy. Therefore, a classical charged particle of finite energy moves in a finite region of the plane perpendicular to the constant magnetic field. This bounded character of motion means that the system is effectively of finite size and translates to the effective dimensional reduction by 2 units in the infrared region in the quantum problem [5]. Since we found the effective universal dimensional reduction $D+1 \rightarrow 1+1$ for fermions on H^D for any D , it is interesting to consider the classical motion in hyperbolic spaces and see whether it has also a bounded character with respect to the coordinates over which the effective dimensional reduction takes place.

The Lagrangian of free classical particle on H^D reads

$$L = \frac{a^2}{x_1^2} (\dot{x}_1^2 + \dot{x}_2^2 + \dots + \dot{x}_D^2). \quad (\text{A6})$$

One can solve classical equations of motion (actually, it is convenient to start with the case of the Lobachevsky plane and the general case $D \geq 3$ can then be deduced by using the spherical symmetry over the x_2, \dots, x_D coordinates) and find that the classical trajectories of motion in H^D (geodesics) have the form

$$z(t) = \ln(x_1(t)) = \ln \frac{v}{A \cosh(vt + C_0)}, \quad x_2(t) = \frac{vC_2}{A^2} \tanh(vt + C_0) + \tilde{C}_2, \dots, \quad x_D(t) = \frac{C_D}{A^2} \tanh(vt + C_0) + \tilde{C}_D \quad (\text{A7})$$

where $A^2 = C_2^2 + C_3^2 + \dots + C_D^2$ and $v, C_0, C_2, \tilde{C}_2, \dots, C_D, \tilde{C}_D$ are arbitrary constants (there are $2D$ of them and, for a particular trajectory, they are fixed by initial conditions) and we introduced also the coordinate $z = \ln x_1$, which is in a certain sense more natural from the viewpoint of metric (A1) because then the spatial interval has the flat space form dz^2 with respect to motion in this coordinate. It is easy to see from (A7) that the motion with respect to z coordinate has the same character as the usual flat space motion except a time interval of order $1/v$. Indeed, it follows from (A7) that the classical particle moves like $z(t) = vt + C$ for $t \ll -\frac{1}{v}$. For $|t| \leq \frac{1}{v}$, its motion differs from the familiar inertial motion in flat space. For $t \gg \frac{1}{v}$, the particle goes back to $-\infty$ (where it started its motion) and its motion has the usual inertial character. On the other hand, motion with respect to x_2, \dots, x_D coordinates has a completely different character. A particle is practically motionless for almost all period of time except the time interval of order $1/v$ when it moves some finite distance. One can calculate the contribution to the spatial interval connected with motion in x_i ($i = \overline{2, D}$) coordinates along geodesics (A7) and find that (unlike motion in x_1 coordinate) it is always finite. For example, the spatial interval connected with motion in x_2 coordinate is equal to

$$l_{x_2} = a \int_{-\infty}^{+\infty} \sqrt{\frac{\dot{x}_2^2}{x_1^2}} dt = \frac{\pi a |C_2|}{A}.$$

Therefore, motion in x_2, \dots, x_D coordinates takes place in a finite region of space defined by initial conditions. At this point similarity with the classical motion of a charged particle in constant magnetic field is transparent, where the particle moves in a bounded region of the plane perpendicular to the magnetic field. For the H^D case, the classical motion has a bounded character in $D - 1$ coordinates. Therefore, it is clear why there is the effective dimensional reduction by 2 units in flat space in a constant magnetic field and the universal reduction $D + 1 \rightarrow 1 + 1$ in hyperbolic spaces H^D in the corresponding quantum problems.

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